Section 1.3 Vector Equations

Vectors in $\mathbb{R}^2$

Definitions

1. A matrix with only one column is called a **column vector** or simply a **vector**. For example,

$$\vec{u} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 0.1 \\ 0.4 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

2. Two vectors in $\mathbb{R}^2$ are **equal** if and only if their corresponding entries are equal. For example,

$$\vec{a} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \text{ are not equal.}$$

3. The set of all vectors with two entries is denoted by $\mathbb{R}^2$ (read "r-two").

4. Given two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$, their sum is the vector $\mathbf{u}+\mathbf{v}$ obtained by adding corresponding entries of $\mathbf{u}$ and $\mathbf{v}$. For example,

$$\vec{u} + \vec{v} = \begin{bmatrix} -3 \\ 2 \end{bmatrix} + \begin{bmatrix} 0.1 \\ 0.4 \end{bmatrix} = \begin{bmatrix} -2.9 \\ 2.4 \end{bmatrix}$$

5. Given a vector $\mathbf{u}$ and a real number c, the scalar multiple of $\mathbf{u}$ by c is the vector $c\mathbf{u}$ obtained by multiplying each entry in $\mathbf{u}$ by c. For example,

$$\text{If } \vec{u} = \begin{bmatrix} -3 \\ 2 \end{bmatrix} \text{ and } c = 4, \text{ then } c\vec{u} = 4 \cdot \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -12 \\ 8 \end{bmatrix}$$

Example 1 Write a vector equation that is equivalent to the given system of equations.

$$\begin{aligned} 4x_1 + x_2 + 3x_3 &= 9 \\ x_1 - 7x_2 - 2x_3 &= 2 \\ 8x_1 + 6x_2 - 5x_3 &= 15 \end{aligned}$$

ANS:

$$\begin{bmatrix} 4x_1 \\ x_1 \\ 8x_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ -7x_2 \\ 6x_2 \end{bmatrix} + \begin{bmatrix} 3x_3 \\ -2x_3 \\ -5x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ 15 \end{bmatrix}$$

or

$$x_1 \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -7 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -2 \\ -5 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ 15 \end{bmatrix}$$

Geometric Descriptions of $\mathbb{R}^2$

We can identify a geometric point (a, b) with the column vector $\begin{bmatrix} a \\ b \end{bmatrix}$.

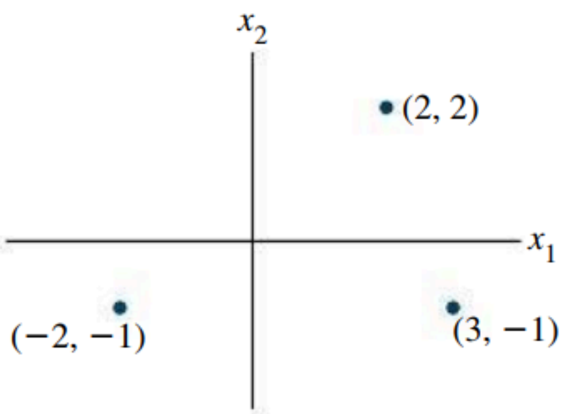


FIGURE 1 Vectors as points.

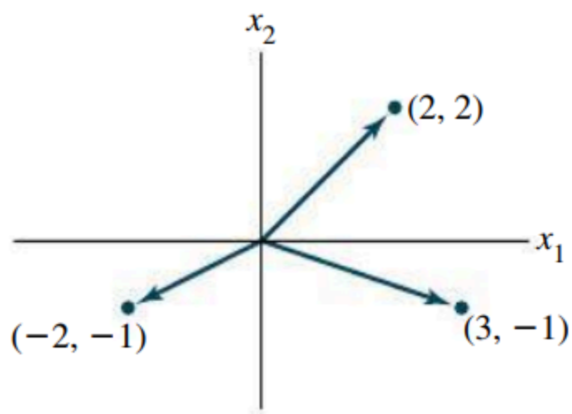


FIGURE 2 Vectors with arrows.

Parallelogram Rule for Addition

If $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$ are represented as points in the plane, then $\mathbf{u}+\mathbf{v}$ corresponds to the fourth vertex of the parallelogram whose other vertices are $\mathbf{u}$, $\mathbf{0}$, and $\mathbf{v}$. See Figure 3.

Given $\vec{u}$, $\vec{v}$ on the x_1-x_2 plane. We have two ways to find $\vec{u}+\vec{v}$: ① $\vec{u}+\vec{v}$ is the diagonal of the parallelogram with two sides $\vec{u}$, $\vec{v}$ passing the origin (0,0)

② starting from the endpt of $\vec{u}$, draw a line parallel to $\vec{v}$ with the same length. The endpt will be $\vec{u}+\vec{v}$.

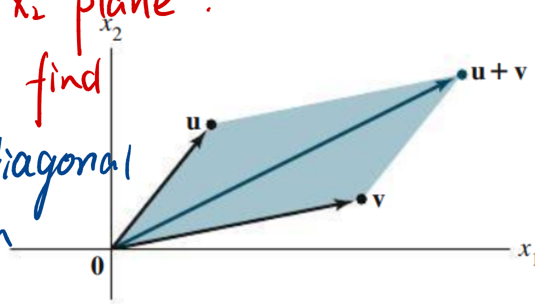
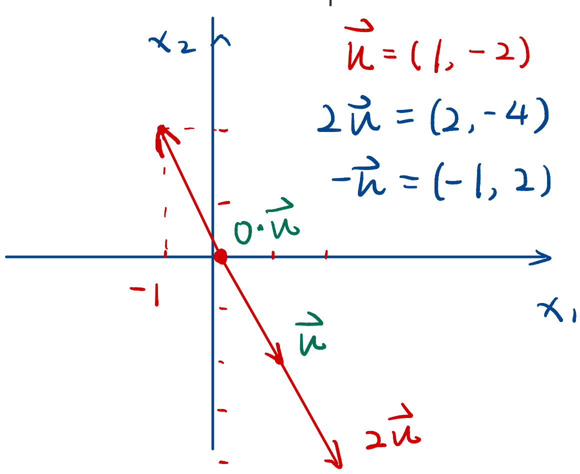
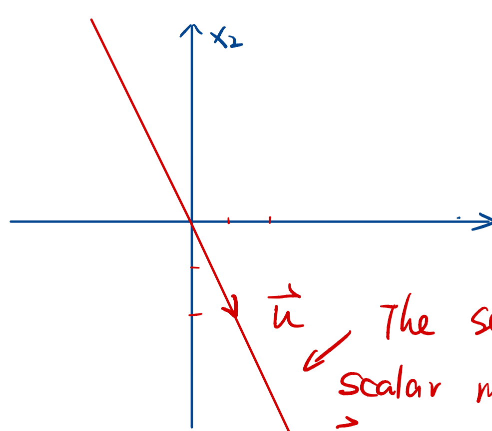


FIGURE 3 The parallelogram rule.

Scalar multiples of a nonzero vector

The set of all scalar multiples of one fixed nonzero vector $\mathbf{u}$ is a line through the origin, $(0, 0)$ and $\mathbf{u}$.





The set of all scalar multiples of $\vec{u}$

Generalization to $\mathbb{R}^3$ and $\mathbb{R}^n$

1. Vectors in $\mathbb{R}^3$ are 3×1 column matrices with three entries.
2. Let n be a positive integer, $\mathbb{R}^n$ denotes the collection of all lists of n real numbers, usually written as $n \times 1$ column matrices, such as

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

Algebraic Properties of $\mathbb{R}^n$

For all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^n$ and all scalars c and d :

$$\vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
- $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$
- $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$, where $-\mathbf{u}$ denotes $(-1)\mathbf{u}$
- $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
- $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
- $c(d\mathbf{u}) = (cd)\mathbf{u}$
- $1\mathbf{u} = \mathbf{u}$

Linear Combinations

Given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ in $\mathbb{R}^n$ and given scalars $c_1, c_2, \ldots, c_p$, the vector $\mathbf{y}$ defined by

$$\mathbf{y} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$$

is called a **linear combination** of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ with weights $c_1, \ldots, c_p$.

Theorem

A vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix} \tag{1}$$

In particular, $\mathbf{b}$ can be generated by a linear combination of $\mathbf{a}_1, \ldots, \mathbf{a}_n$ if and only if there exists a solution to the linear system corresponding to the matrix (1).

$\Updownarrow$

$b \in \mathrm{Span}\{a_1, \cdots, a_n\}$ (by the def of span$\{a_1, \cdots, a_n\}$)

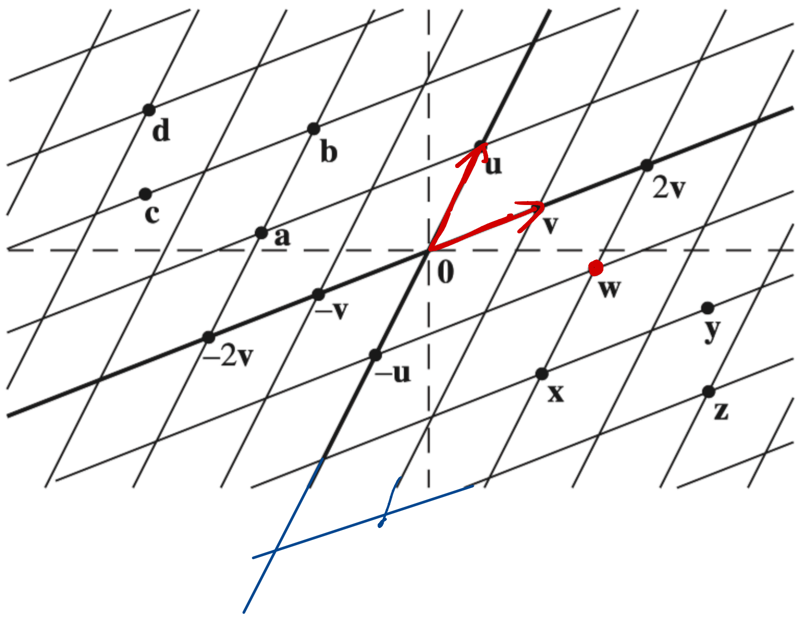
Definition: Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$

If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is denoted by Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ and is called the subset of $\mathbb{R}^n$ spanned (or generated) by $\mathbf{v}_1, \ldots, \mathbf{v}_p$. That is, $\mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

with $c_1, \ldots, c_p$ scalars.

Example 2 Use the accompanying figure to write vectors $\mathbf{w}, \mathbf{x}, \mathbf{y}$, and $\mathbf{z}$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$. Is every vector in $\mathbb{R}^2$ a linear combination of $\mathbf{u}$ and $\mathbf{v}$?



$\vec{w} = -\vec{u} + 2\vec{v}$

To reach $\vec{w}$ from the origin. travel -1 units in the $\vec{u}$ direction then travel 2 units in the $\vec{v}$ direction.

Similarly.

$\vec{x} = -2\vec{u} + 2\vec{v}$

$\vec{y} = -2\vec{u} + 3.5\vec{v}$

$z = -3\vec{u} + 4\vec{v}$

For $\vec{y}$, we can also start from x. and travel 1.5 units in the direction of $\vec{v}$, So $\vec{y} = \vec{x} + 1.5\vec{v} = -2\vec{u} + 2\vec{v} + 1.5\vec{v} = -2\vec{u} + 3.5\vec{v}$

Example 3 Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix}$.

Determine if $\mathbf{b}$ is a linear combination of $\mathbf{a}_1, \mathbf{a}_2$, and $\mathbf{a}_3$. That is, determine whether weights x_1, x_2 and x_3 exist such that ($\Leftrightarrow$ if $\vec{b}$ is in Span$\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$)

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$$

ANS: By the previous Thm, the vector eqn.

$$x_1 \underbrace{\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}}_{\vec{a}_1} + x_2 \underbrace{\begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix}}_{\vec{a}_2} + x_3 \underbrace{\begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}}_{\vec{a}_3} = \underbrace{\begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix}}_{\vec{b}}$$

has the same solution set as the linear system whose augmented matrix is.

$$M = \left[\begin{array}{ccc|c} 1 & 0 & 2 & -5 \\ -2 & 5 & 0 & 11 \\ 2 & 5 & 8 & -7 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & -5 \\ 0 & 5 & 4 & 1 \\ 0 & 5 & 4 & 3 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & -5 \\ 0 & 5 & 4 & 1 \\ 0 & 0 & 0 & -2 \end{array}\right] \longrightarrow \text{This means } 0 = -2 \text{ (impossible!)}$$

The corresponding linear system has no solution

Thus $\vec{b}$ is not a linear combination of $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3$

(or $\vec{b}$ is not in Span$\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$)

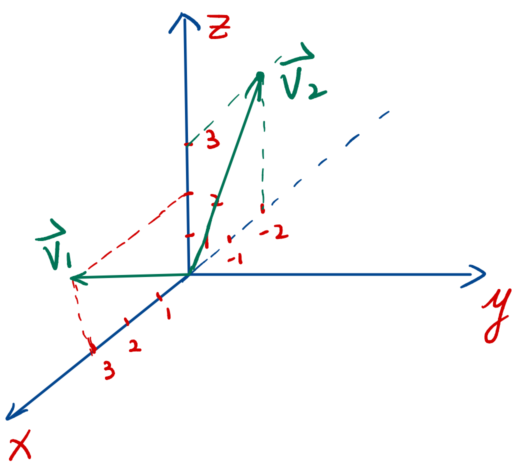
A Geometric Description of Span$\{\mathbf{v}\}$ and Span$\{\mathbf{u},\mathbf{v}\}$

by def. Span$\{v\}$ is a collection of $c\vec{v}$, for all $c \in \mathbb{R}$.

1. Let $\mathbf{v}$ be a nonzero vector in $\mathbb{R}^3$, $\text{Span}\{\mathbf{v}\}$ is the set of points on the line in $\mathbb{R}^3$ through $\mathbf{v}$ and $\mathbf{0}$.
2. Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in $\mathbb{R}^3$, and $\mathbf{v}$ is not a multiple of $\mathbf{u}$, then $\cancel{\{\mathbf{u},\mathbf{v}\}}$ Span$\{\vec{u},\vec{v}\}$ is the plane in $\mathbb{R}^3$ that contains $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{0}$.

Example 4 Give a geometric description of $\text{Span}\{\mathbf{v}_1,\mathbf{v}_2\}$ if

(i) $\mathbf{v}_1 = \begin{bmatrix}3\\0\\2\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}-2\\0\\3\end{bmatrix}$



Notice that

① $\vec{v}_1$ $\vec{v}_2$ are not scalar multiples of each other.

② Both $\vec{v}_1, \vec{v}_2$ has 0 on the y coordinate.

So they are on the xz plane

Thus. Span$\{\vec{v}_1\ \vec{v}_2\}$ is the xz plane.

(ii) $\mathbf{v}_1 = \begin{bmatrix}4\\1\\-3\end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix}12\\3\\-9\end{bmatrix}$

Notice that $\vec{v}_2 = 3\vec{v}_1$

Thus any linear combination of $\vec{v}_1$ and $\vec{v}_2$ is a multiple of $\vec{v}_1$. since

$$c_1\vec{v}_1 + c_2\vec{v}_2 = c_1\vec{v}_1 + c_2\cdot 3\vec{v}_1 = (3c_2+c_1)\vec{v}_1$$

So Span$\{\vec{v}_1, \vec{v}_2\}$ is the set of points on the line through $\vec{v}_1$ and $(0,0,0)$

The following two questions are left as exercises. I will provide the complete notes for solving them after the lecture.

Exercise 5 Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -3 \\ 1 \\ 8 \end{bmatrix}$, and $\mathbf{y} = \begin{bmatrix} h \\ -5 \\ -3 \end{bmatrix}$. For what value(s) of h is $\mathbf{y}$ in the plane spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$?

ANS: By the definition of $\operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, we know that $\mathbf{y}$ is in the plane spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$ if and only if the vector equation $\mathbf{y} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2$ has solution(s). The corresponding augmented matrix is:

$$[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{y}] = \begin{bmatrix} 1 & -3 & h \\ 0 & 1 & -5 \\ -2 & 8 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & h \\ 0 & 1 & -5 \\ 0 & 2 & -3+2h \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & h \\ 0 & 1 & -5 \\ 0 & 0 & 7+2h \end{bmatrix}.$$

Thus vector $\mathbf{y}$ is in $\operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ when $7+2h$ is zero, that is, when $h = -7/2$.

Exercise 6 Let $A = \begin{bmatrix} 2 & 0 & 6 \\ -1 & 8 & 5 \\ 1 & -2 & 1 \end{bmatrix}$, let $\mathbf{b} = \begin{bmatrix} 10 \\ 3 \\ 3 \end{bmatrix}$, and let W be the set of all linear combinations of the columns of A.

a. Is $\mathbf{b}$ in W ?

b. Show that the third column of A is in W.

ANS:

a. Denote the columns of A by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$. Then $W = \operatorname{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.

Note that $\mathbf{b}$ is in W if and only if the vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$ has solution(s). We check the corresponding augmented matrix:

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{b}] = \begin{bmatrix} 2 & 0 & 6 & 10 \\ -1 & 8 & 5 & 3 \\ 1 & -2 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 5 \\ -1 & 8 & 5 & 3 \\ 1 & -2 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 8 & 8 & 8 \\ 0 & -2 & -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 8 & 8 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So the system has at least one solution (in fact, infinitely many solutions).

Thus $\mathbf{b}$ is a linear combination of the columns of A, that is, $\mathbf{b}$ is in W.

b. The third column of A is in W because $\mathbf{a}_3 = 0 \cdot \mathbf{a}_1 + 0 \cdot \mathbf{a}_2 + 1 \cdot \mathbf{a}_3$.